# A COMPARATIVE ANALYSIS: TIME VS. FREQUENCY DOMAIN DEFINITIONS OF THE FATEMI-SOCIE CRITERION

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## ABSTRACT

The article aims to adapt the Fatemi-Socie criterion (FSC) to the frequency domain and compare it with the time domain approach to evaluate fatigue life. Key challenges include determining the critical plane position, replacing cycle counting methods with models estimating rainflow amplitude distributions, and defining a probability density function for strain amplitude and maximum normal stress. The methodology involves defining the maximum variance method to locate the critical plane and using moments of PSD to establish a two-dimensional probability distribution function for shear strain and normal stress. The study verifies the proposed method against known solutions and applies it to random load cases to demonstrate its effectiveness. Results indicate a strong correlation between fatigue life predictions in both domains, confirming the robustness and versatility of the FSC. The findings suggest that the proposed frequency domain approach, utilizing statistical models, can reliably extend FSC applications to complex loading scenarios, offering a valuable tool for fatigue analysis in engineering applications.

## KEYWORDS

Multiaxial fatigue, Fatemi-Socie criterion, vibration fatigue, frequency domain, power spectral density

## INTRODUCTION

In the realm of multiaxial fatigue analysis, the Fatemi-Socie criterion (FSC) stands as a well-established and validated tool, having undergone rigorous experimental scrutiny with consistently favorable outcomes across a wide spectrum of materials [1]. Prior evaluations have predominantly involved comparisons under constant amplitude, both in proportional and non-proportional loading scenarios, with less frequent exploration of variable amplitude or random loadings [2]. Notably, there has been a gap in employing this criterion, particularly in the context of frequency domain and algorithms utilizing the power spectral density function. In this study, the authors introduce a comprehensive adaptation of the Fatemi-Socie criterion to the frequency domain and compare the resulting fatigue assessments with those obtained in the time domain. The comparative analysis extends to critical plane identification, critical plane amplitudes and maxima analysis, and overall fatigue life determination. The findings reveal a compelling similarity in fatigue life predictions between computational algorithms operating in both the time and frequency domains, highlighting the criterion's versatility and robustness.

## FATEMI-SOCIE CRITERION

Fatemi and Socie have presented their critical plane based approach to multiaxial fatigue in 1988 [3]. They assume that, for a given fatigue life $N_f$, the fatigue parameter $P_{FS}$ expressed by

$$P_{FS} = \gamma_{a,max}\left(1 + k\frac{\sigma_{n,max}}{\sigma_y}\right) = const. \tag{1}$$

should be constant for various configurations of loading. Two main quantities are responsible for material fatigue: the maximum shear strain amplitude $\gamma_{a,max}$ and the maximum normal stress $\sigma_{n,max}$ evaluated on the plane with the maximum fatigue parameter $P_{FS}$. This proposed criterion belongs to the group of strain criteria, because the maximum normal stress $\sigma_{n,max}$ is divided by the yield strength of the material $\sigma_y$ which results in the expression in parentheses in Eq. (1) being dimensionless. A second issue supporting this statement is that the parameter $P_{FS}$ should be compared to shear strain fatigue characteristic. This statement leads also to the equation

$$P_{FS} = \frac{\tau'_f}{G}(2N_f)^{b_0} + \gamma'_f(2N_f)^{c_0} \tag{2}$$

which is used to evaluate the fatigue life. However it is not often that the results from uniaxial strain controlled torsional tests, performed on tubular unnotched specimens, are available for the engineer. In view of this, Fatemi and Kurath, according to the deliberation presented in the appendix of [4], have derived the following equation

$$P_{FS} = \left[(1+\nu_E)\frac{\sigma'_f}{E}(2N_f)^b + (1+\nu_P)\varepsilon'_f(2N_f)^c\right]\cdot\left[1 + k\frac{\sigma'_f}{2\sigma_y}(2N_f)^b\right] \tag{3}$$

which provides the possibility of performing fatigue life assessments using the $P_{FS}$ parameter and axial strain Woehler curve, obtained with unnotched specimens under strain control. The FSC in its basic form, Eq. (1), is defined for a multiaxial constant amplitude load. Such a load is not often encountered in service load histories, where variable-amplitude or even random loads dominate. Hence, many scientific works have been devoted to adapting the FSC to the case of loads with variable amplitude and random loads. Shamsaei et al. [5] performed a number of fatigue tests for amplitude and block loads in which the blocks had different multiaxial load configurations. In this case, the resulting damage for all blocks load was determined in accordance with Palmgren-Miner's damage accumulation rule by summing up the partial damages for each load configuration, i.e. each block. Following this, damage summation can be performed for each pair of strain amplitude and maximum normal stress, provided the pairs are identified by the multiaxial rainflow procedure [6].

## FREQUENCY DOMAIN DEFINITION

The difficulty in establishing multiaxial fatigue damage criteria in the frequency domain lies in the reliance on statistical methods rather than deterministic values. This requires the utilisation of statistics base on moments of PSD and probability distribution estimation instead of fixed values coming from measurements or time histories of stress and strain. Moreover, referring to the Fatemi-Socie criterion, it can be noted that its definition is made for a constant-amplitude load with a phase shift. Therefore, when developing a criterion in the frequency domain, attention should be paid to later solutions that allowed the use of the criterion in the state of stress and strain with variable amplitude loading condition. Such a solution is proposed by Bannantine and Socie in publication [7], [8] based on critical plane concept and rainflow cycle counting technics. In this proposal, the main channel for counting cycles is the time history of shear strain determined in the critical plane, then for each shear strain cycle the normal stress course is searched to determine its maximum value during the time of the cycle occurrence. As a result, numerous pairs of strain amplitude and maximum stress values are obtained. As a consequence of obtaining many pairs of parameters, the final value of fatigue damage for one position of the critical plane is obtained by summing the damage in accordance with Palmgen-Miner hypothesis of damage accumulation. This is consistent with the idea of the FS parameter defined for constant amplitude loading where only one pair of such parameters are noted. Summarizing the above, it is possible to enumerate the challenges that need to be addressed to define this criterion in the frequency domain using only the PSD matrix of the strain and stress. These are:

- determination of the critical plane position for which the maximum damage method or the maximum variance method can be applied,
- replacing the cycle counting method with models for estimating the rainflow amplitude distribution known in the spectral method,
- replacing the indication of the maximum stress for each strain rainflow cycle with the distribution of maximum stresses for the strain amplitude interval.

**Setting the critical plane position**

In the spectral method, analogous to the time domain approach, the maximum damage method and the maximum variation method can be employed to identify the critical plane. In the maximum damage method, damage is evaluated in multiple potential orientations of the critical plane, and the orientation yielding the highest damage parameter is selected. This method typically demands substantial computational effort and is therefore infrequently used in practical applications. Conversely, the maximum variation method seeks to maximize the parameter that characterizes damage, specifically the Fatemi-Socie parameter. To apply the maximum variation method, the parameter $P_{FS}$ must be defined in terms of the variance of the relevant quantities

$$P_{FS} = \sqrt{2v_\gamma}\left(1 + k\frac{\sigma_m + \sqrt{v_\sigma}F(N)}{\sigma_y}\right) \tag{4}$$

where $v_\gamma$ is the variance of the maximum shear strain acting on critical plane, $v_\sigma$ and $\sigma_m$ are the variance and global mean of the stress normal to the critical plane, and

$$F(N) = \sqrt{2\ln(N)} + \frac{0.5772}{\sqrt{2\ln(N)}} \tag{5}$$

is the function for maximum value in random process according Davenport [9]. When searching all planes, the number of cycles $N$ is unknown because only the variance is used. It is recommended to set $N = 1e6$ and keep this value constant when searching for the critical plane.

**Replacing the rainflow counting with join probability density function**

To accurately formulate the FSC in the frequency domain, it is essential to understand the summation of fatigue damage in this criterion which involves analysing numerous pairs of shear strain amplitude $\gamma_a$ and maximum normal stress values $\sigma_{n,max}$. The strain amplitude is determined using the rainflow method, and the maximum stress value is recorded during the load cycle corresponding to this amplitude. All these measurements pertain to a single fixed position of the critical plane.

Let us first analyse the distribution of shear strain amplitudes, which, in the spectral method, can be described by models that estimate amplitudes from a random process. Numerous proposals for these models exist, with the most common being the Dirlik model [10]

$$p_\gamma(\gamma_a) = \frac{1}{\sqrt{m_{\gamma 0}}}\left[\frac{K_1}{K_4}e^{-\frac{Z}{K_4}} + \frac{K_2 Z}{R^2}e^{-\frac{Z^2}{2R^2}} + K_3 Z e^{-\frac{Z^2}{2}}\right] \tag{6}$$

where

$$Z = \frac{\gamma_a}{\sqrt{m_{\gamma 0}}} \qquad K_1 = \frac{2(x_m - I^2)}{1 + I^2} \qquad K_2 = \frac{1 - I - K_1 + K_1^2}{1 - R} \qquad x_m = \frac{m_{\gamma 1}}{m_{\gamma 0}}\sqrt{\frac{m_{\gamma 2}}{m_{\gamma 4}}}$$

$$K_3 = 1 - K_1 - K_2 \qquad R = \frac{I - x_m - K_1^2}{1 - I - K_1 + K_1^2} \qquad Q = \frac{5(I - K_3 + K_2 R)}{4K_1} \qquad I = \frac{m_{\gamma 2}}{\sqrt{m_{\gamma 0} m_{\gamma 4}}}$$

are coefficients computed directly from shear strain PSD moments

$$m_{\gamma k} = \int_0^\infty G_\gamma(f) f^k df \quad \text{for} \quad k = [0,1, \ldots 4] \tag{7}$$

In the case of a narrowband random load with fully correlated components, a similar phenomenon is observed as with an in-phase constant amplitude load, where local maxima for shear strain and normal stress occur simultaneously. This results in one value of normal stress corresponding to each value of shear strain amplitude. However, for a random load with a wide load spectrum and varying degrees of correlation between components, a distribution of maximum stress is expected for a given shear strain amplitude. As a result, a two-dimensional matrix of shear strain amplitude and maximum normal stress values is obtained. This matrix, evaluated in time domain using two stress components $\sigma_{xx}(t)$, $\tau_{xy}(t)$ and rainflow, is shown on Fig. 1 for (a) narrow-band loading and $r_{\sigma xx,\tau xy} = 1.0$, (b) narrow-band loading and $r_{\sigma xx,\tau xy} = 0.0$; (c) broadband loading and $r_{\sigma xx,\tau xy} = 1.0$; (d) broadband loading and $r_{\sigma xx,\tau xy} = 0.0$. The two-dimensional probability distribution is influenced by the spectrum width and the correlation between shear strain amplitude and maximum normal stress. In the extreme case of a narrow-band correlated load, Fig. 1(a), this distribution becomes near flat, extending from the origin (0, 0) to the maximum values of stress and strain amplitude. It can be assumed that this distribution can be described by Dirlik's amplitude distribution, Eq. (6). For each strain amplitude, a normal distribution of extreme values is observed with accordance to the exceedance theory. Following that the join probability density function can be formulated as follow

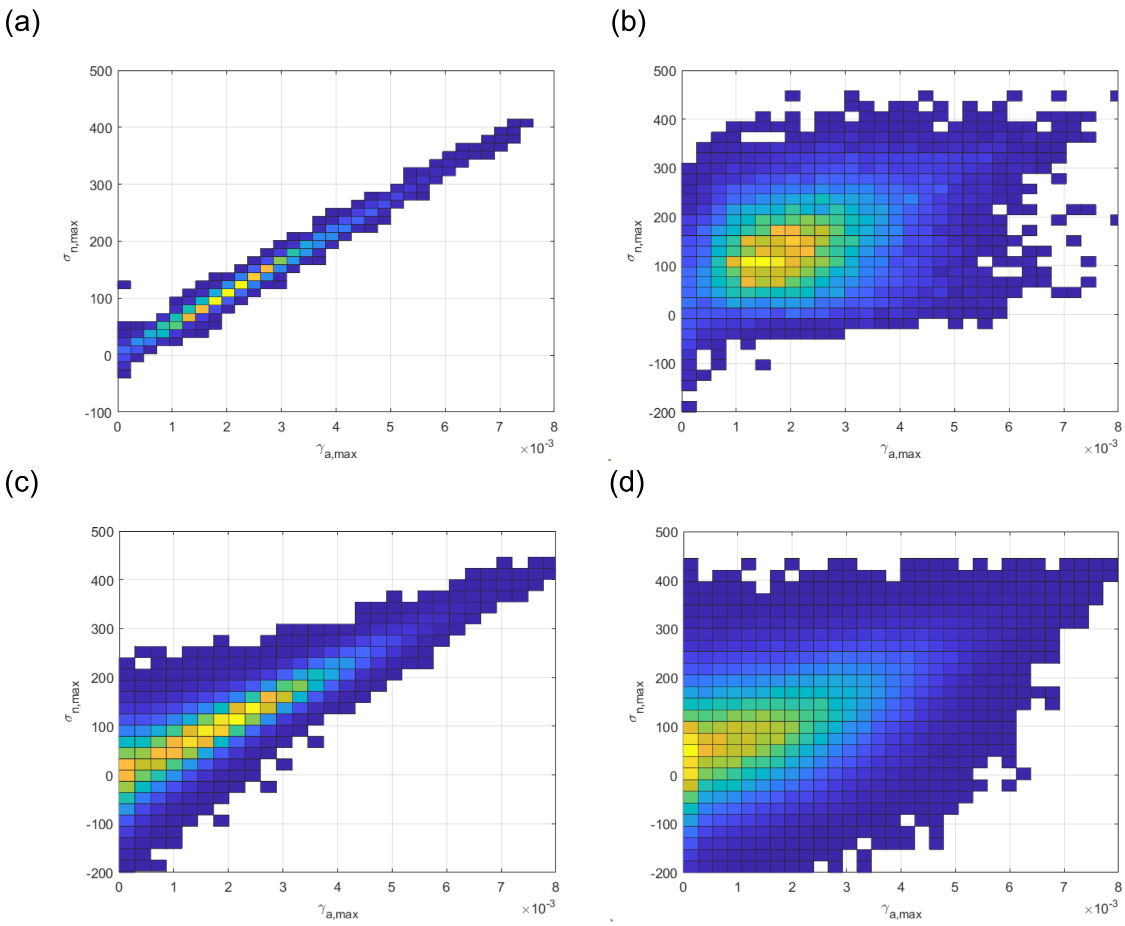


Fig. 1: Distribution of the strain amplitude and maximum stress pairs on critical plane obtained by rainflow method for (a) narrowband loading and $r_{\sigma xx,\tau xy} = 1.0$; (b) narrowband loading and $r_{\sigma xx,\tau xy} = 0.0$; (c) broadband loading and $r_{\sigma xx,\tau xy} = 1.0$; (d) broadband loading and $r_{\sigma xx,\tau xy} = 0.0$;

$$p_{\gamma\sigma}(\gamma_a, \sigma_{n,max}) = \phi(\sigma_{n,max}; \phi_\mu, \phi_\sigma) p_D(\gamma_a; m_\gamma) \tag{8}$$

where $\phi(\sigma_{n,max}; \phi_\mu, \phi_\sigma)$ is the normal distribution and $p_D(\gamma_a; m_\gamma)$ is the rainflow amplitude distribution according Dirlik. The mean value and variance in the normal distribution evolve as functions of the spectrum width and correlation. The following proposals for these values have been formulated:

$$\phi_\mu = \sqrt{\frac{m_{\sigma 0}}{m_{\gamma 0}}} \gamma_a + \sigma_g \tag{9}$$

$$\phi_\sigma = \left[ 1 - |r_{\gamma\sigma}| + \sqrt{1 - \left(\frac{E0}{EP}\right)^2} \right] \sqrt{m_{\sigma 0}} \left( 1 - \frac{\gamma_a}{\sqrt{m_{\gamma 0}} F(N)} \right) \tag{10}$$

where: $m_{\sigma 0}$ is the zero moment of the stress PSD, $\sigma_g$ is the global mean stress, $|r_{\gamma\sigma}|$ is the absolute value of correlation coefficient between shear strain and normal stress, $E0$ and $EP$ are expected zero crossing and peaks of stress. Using the defined two-dimensional probability density in Eq. (8), the Fig. 2 was created, showing the distributions estimated solely based on the power spectral density of shear strain and normal stress on the critical plane.

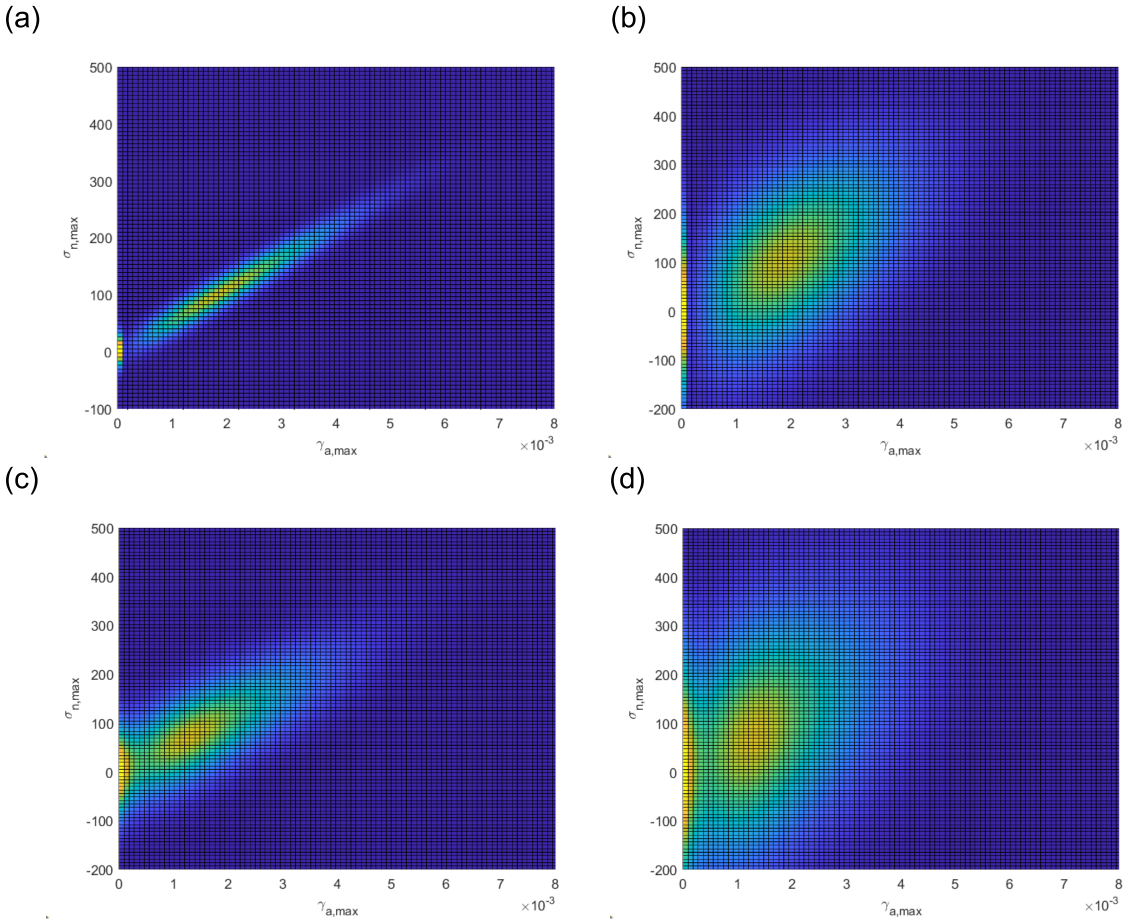


Fig. 2: Distribution of the strain amplitude and maximum stress pairs on critical plane according Eq. (6), in accordance with the data presented on Fig. 1, (a) narrowband loading and $r_{\sigma xx,\tau xy} = 1.0$; (b) narrowband loading and $r_{\sigma xx,\tau xy} = 0.0$; (c) broadband loading and $r_{\sigma xx,\tau xy} = 1.0$; (d) broadband loading and $r_{\sigma xx,\tau xy} = 0.0$.

One of the main computational challenges is determining the correlation coefficient $r_{\gamma\sigma}$ between the shear strain and the normal stress evaluated on critical plane. This parameter is crucial because uncorrelated signals exhibit a significant spread in the normal stress distribution for a given shear strain amplitude. Assessing this parameter involves determining the cross power spectral density function between the normal and shear stress on the critical plane. It is assumed that the correlation coefficient between these values is the same as the shear stress is proportional to the shear strain and can be evaluated as follow

$$r_{\gamma\sigma} = r_{n\tau} = corr\left\{\vec{n}\boldsymbol{\sigma}(t)\vec{n}^{'}, \vec{\tau}\boldsymbol{\sigma}(t)\vec{\tau}^{'}\right\} \tag{11}$$

where $\boldsymbol{\sigma}(t)$ is the stress tensor and the vectors $\vec{n}$ and $\vec{\tau}$ are presented on Fig. 3. In the frequency domain, the correlation coefficient can be computed as follows

$$r_{n\tau} = \sqrt{\frac{\int |G_{n\tau}(f)|^2 df}{\left(\int G_{nn}(f)df\right)\left(\int G_{\tau\tau}(f)df\right)}} \tag{12}$$

where $G_{n\tau}(f)$ is the cross-spectral density between the normal and shear stress and $G_{nn}(f)$ and $G_{\tau\tau}(f)$ are the PSD of the normal stress and shear stress, respectively. Obtaining the power spectral density values in the critical plane is a well-established procedure, detailed in publications by Mršnik et al. [11] and Gao et al. [12], among others.

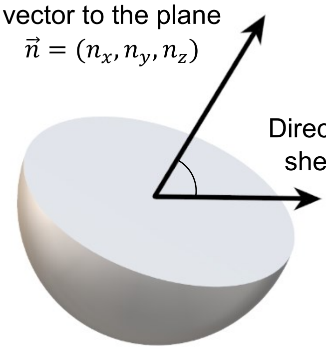


Fig. 3: Vectors $\vec{n}$ and $\vec{\tau}$ defining the critical plane and the direction of the shear strain.

**CONCLUSION AND REMARKS**

1. To define the multiaxial Fatemi-Socie criterion in the frequency domain, one must first express the $P_{FS}$ parameter in terms of covariance matrixes of strain and stress. This allows the use of the maximum variance method to determine the critical plane. This approach involves calculating the equivalent shear strain amplitude expressed as the square root of the variance multiplied by two and the maximum value of normal stress, based on stress covariance matrix in accordance with Davenport's theory, Eq. (4).

2. It was observed that for random processes, the Fatemi-Socie criterion necessitates determining pairs of shear strain amplitudes and the maximum values of normal stress on the critical plane, given a fixed orientation of this plane. In the time domain, this is achieved using the rainflow algorithm. To apply the Fatemi-Socie criterion in the frequency domain, a two-dimensional probability distribution was defined by combining the normal and Dirlik distribution, presented as a Eq. (8). The proposed probability distribution uses only parameters determined from the stress and strain PSD function evaluated at critical plane, which allows it to be used without moving to the time domain.

3. The proposed probability distribution, Eq. (8), describes the occurrence of pairs of strain amplitudes and maximum stress values in a manner similar to those obtained in the time domain using the rainflow algorithm.

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